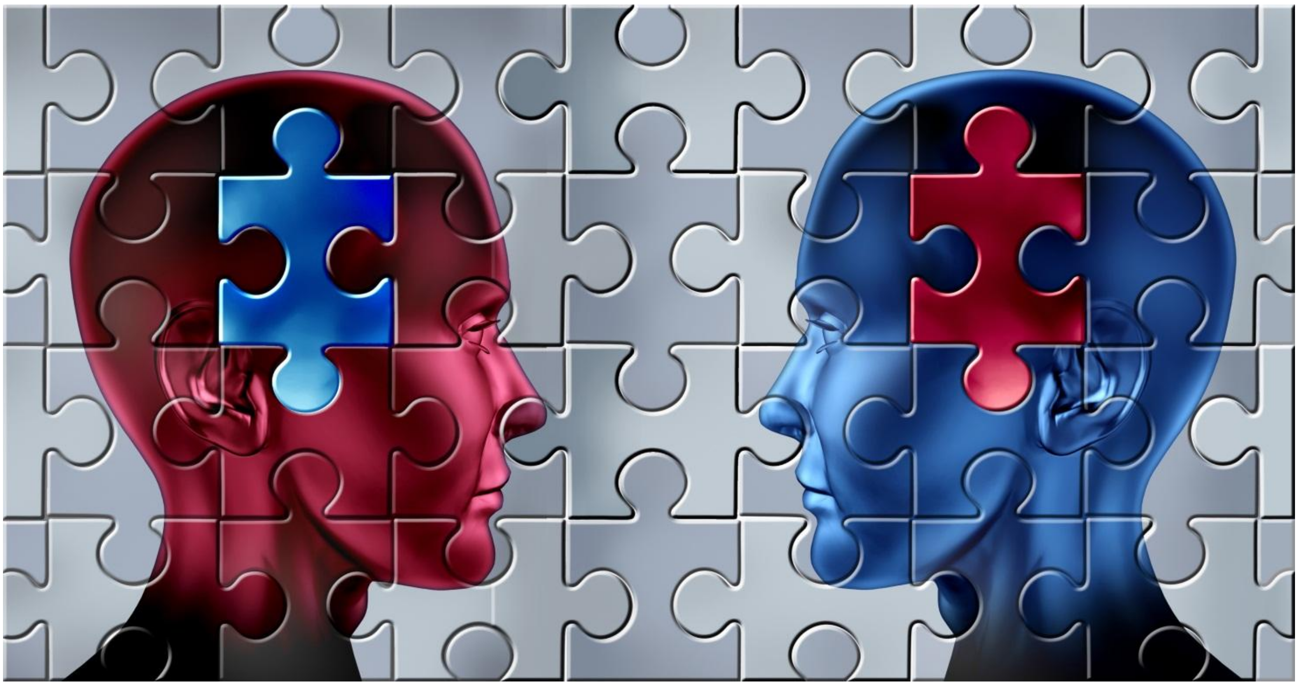

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Abstract

Actual motion of a vibrating guitar string is a superposition of many possible shapes (modes) in which the string could vibrate. Each of these modes has a corresponding frequency, and the lowest frequency is associated with a shape idealised as a single wave, referred to as the fundamental mode. The other contributing modes, each with their own progressively higher frequency, are referred to as overtones, or harmonics. By attaching a string to a medium (a sound board) capable of a response to the vibrating string, sound waves are generated. The sound heard is dominated by the fundamental mode, 'coloured' by contributions from the overtones, as explained by the classical theory of vibration. The classical theory, however, assumes that the string tension remains constant during vibration, and this cannot be strictly true; when considering just the fundamental mode, string tension will reach two maximum changes, as it oscillates up and down. These changes, occurring twice during the fundamental period match the frequency of the octave higher, 1st overtone. It is therefore plausible to think that the changing tension effect, through increased force on the bridge and, therefore, greater sound board deflection, could be amplifying the colouring effect of (at least) the 1st overtone.

In this paper, we examine the possible influence of string tension variation on tonal response of a classical guitar. We use a perturbation model based on the classical result for a string in general vibration in conjunction with a novel method of assessment of plucking force that incorporates the engineering concept of geometric stiffness, to assess the magnitude of the normal force exerted by the string on the bridge. The results of our model show that the effect of tension variation is significantly smaller than that due to the installed initial static tension, and affects predominantly the force contribution arising from the fundamental mode. We, therefore, conclude that string tension variation does not contribute significantly to tonal response.

Keywords: guitar tonal response; string vibration; perturbation model; overtones; tension modulation; geometric stiffness.

Introduction

It is sometimes remarked in classical guitar circles that, in spite of the relatively low tuning of the six-string tenor guitar, viz., E2, A2,D3,G3,B3,E4 or E,A,D,g,b,e′, the overall effect in solo performance is of a rather higher pitched instrument. This impression can be considered to be predicted by the classical theory of a vibrating, uniform, thin, laterally flexible string with rigid end attachments. It is noted that this classical theory is universally believed to give a satisfactory, 'first order' model of string behaviour, in terms of predicting vibrational modes and frequencies. Taylor (1978) applied this classical background to give a physical basis for tonal control on a classical guitar. The perceived sound of a guitar or, indeed, any instrument, is a mixture of tones stemming from the 'modes' that are excited by the relevant playing action; in the case of a guitar: plucking a string. With each mode is associated a frequency that 'colours' the sound. Taylor used no overt mathematics, but graphs illustrating how energy input to the overtones is controlled by: i) plucking position along the string, ii) plucking mechanisms, i.e., the use of nails or flesh tips. He showed that, as the string is plucked with a point force, and the position of plucking is progressively moved closer to an end, energy input to the overtones is increased at the expense of the fundamental. Further, the use of nails, modelled as a point force, as against flesh tips, modelled as a short wedge plectrum, showed that nails have a noticeably greater ability to energise overtones.

Figure 1 shows a classical guitar with all its relevant parts identified.

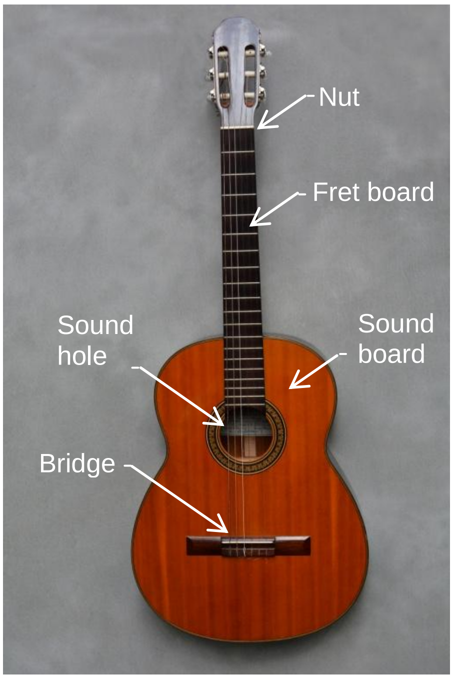
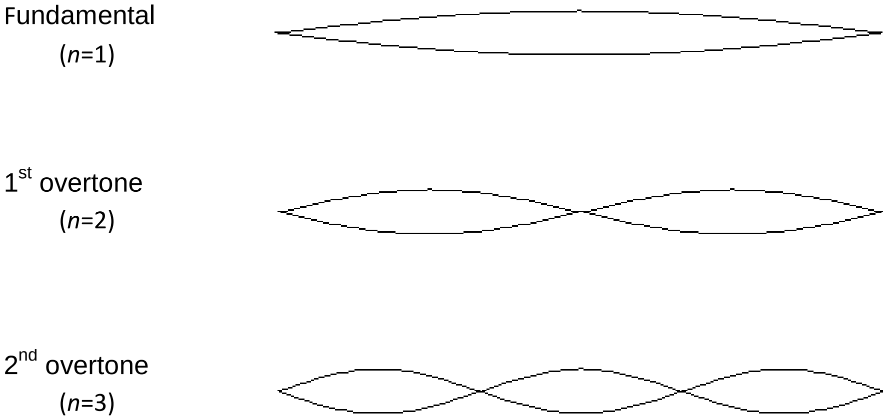


Fig. 1. A classical guitar (Mexican style)



A guitar string vibrates in superposition of a number of modes, or shapes, which include: fundamental (1st) mode, 2nd, 3rd, and higher level modes. The fundamental mode produces the lowest tone. The 2nd mode (1st overtone), and correspondingly, the higher modes are aligned with higher frequencies (Fig. 2). The first overtone is at double the frequency of the fundamental, and, in musical terms, is referred to as 'an octave higher'.

Fig.2. First three modes of vibration and the corresponding overtones

Normal plucking position for a guitar (Fig. 3) varies from just over the sound hole to one on the bridge side of the sound hole. More extreme positions magnify the subtle effects achieved by small changes of plucking position/mixing nail and flesh tip. Thus, playing over the fret board, i.e., playing 'sul tasto', produces a dolce/deeper sound, as the fundamental is emphasised and the overtones are less energised.

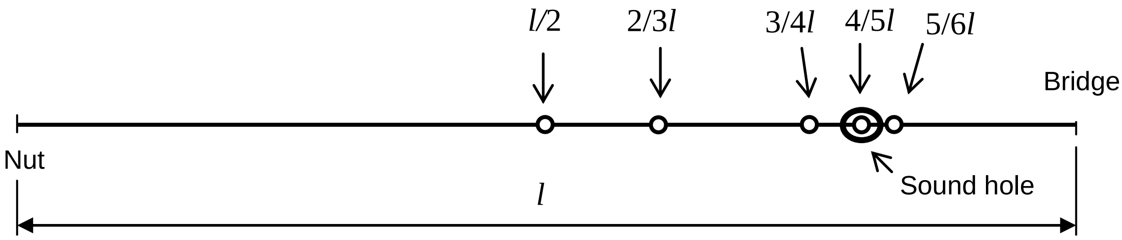


Fig. 3. Typical plucking positions

- $l/2$ - at the 12th fret $\rightarrow$ this emphasises the fundamental mode of the open string
- $2/3l$ - for playing 'sul tasto'- over the fret board. As one is often playing in the first few frets, sul tasto corresponds to plucking at about the half length of the fretted string
- $3/4l$- just on the nut side of the sound hole
- $4/5l$- at the sound hole
- $5/6l$ – on the bridge side of the sound hole

Common right hand positions

As already noted (Taylor 1978), plucking progressively near the bridge, i.e., playing 'ponticello', changes the tone by putting greater energy into the overtones. The effect is a much higher, thinner sound, as the fundamental is weakened and the overtones strengthened. It is clear that the classical theory does give an indication as to why a guitar can give an impression of a somewhat higher pitched instrument. This classical theory is given in many texts for undergraduate physicists and mathematicians covering waves and vibrations. The classic references would be Morse (1948), Ramsey (1949) and earlier editions, Coulson (1941) and subsequent editions.

The crucial point to explore beyond the classical theory's assumption of small displacements and slopes, with its consequential opportunities for linearising the resulting equations, is just the relaxation of this assumption. Work on string vibration, where tension changes are appreciable, has fallen into two classes: i) that dealing with the actual equations governing large vibrations, and ii) that which adopt a perturbation approach. In the perturbation approach, which is used in the work reported in this paper, the length of the string during vibration is taken to be that of the profile predicted by the classical theory.

Interestingly, early examples of modelling, where coupled longitudinal and lateral motions are jointly considered, are given in Ramsey (1949, Chap. X1, problems 35 and 36), where, in 36, longitudinal motion is not small, but lateral motion is. However, it seems that Carrier (1945) was the first publication to derive the equations of motion of a string rigidly held at both ends that are valid at large amplitude. A very different approach is that of Antman (1980), whose work is overtly critical of the majority of derivations of string wave equations, including constraint to motion in one plane. The final stages of the paper are concerned with the analytical problems involved in approximating the governing non-linear equations. Murthy and Ramakrishna (1964) modelled a problem of a string excited at, or near, a resonant frequency. Their work confirmed that, near resonance, string response is governed more by variation in tension than damping. The foregoing are of type i) in their approach. The following papers are of type ii).

The study by Legge and Fletcher (1984) concentrated on a well-known prediction of the classical theory that if a string is plucked at ½ its length, the 2nd mode and all its integer multiples are missing, and similarly for plucking at $1/n^{th}$ of the length. The authors noted that in musical instruments, where one string end does not have a rigid attachment, the missing mode phenomenon is no longer strictly valid. Their model was based on the one-dimensional wave equation, but included damping.

A valuable overview of non-linear effects in a wide range of musical instruments is given by Fletcher (1999), in which the coverage is essentially similar to that of the jointly authored text by Fletcher and Rossing (1998). On the dynamics of plucked strings, he recommends the inclusion of bending elasticity, but indicates that this is not important for the nylon strings of a classical guitar, due to their low Young's modulus. For the estimation of string tension variation during vibration, he indicates the use of a vibrating string profile predicted by the classical linear theory, i.e., a perturbation model, as used in this paper. However, there is no discussion of the influence of tension variation on overtone response.

Tolonen *et al.* (2000), and Band (2009), simulated the phenomenon of 'pitch glide', which occurs when a string is given a large lateral displacement and released. During vibration, the string motion is damped by air resistance and the string's internal dissipation mechanism. Damping reduces the amplitude of vibration and, consequently, the tension falls, as does frequency. The audible effect is a perceptible 'wow'; a more noticeable characteristic of guitars with steel rather than nylon strings. With virtually no discussion of the underlying physics, Tolonen *et al.* simulated pitch glide using a digital wave guide and the main content is concerned with showing how the various filters are generated. Band does review the relevant underlying physics in some detail.

None of the work reviewed above produces a model relating tension variation in the string to a possible enhancement of overtone response.

The classical theory assumes constant tension, because the lateral displacements of strings used in musical instruments are small compared to their overall length. However, strings have longitudinal elasticity, and it can be shown that, while change in length may be small, the increase in tension may be noticeable. Lewis (2003) made a similar observation when discussing the response of tension cable structures to lateral displacements.

If we consider just the fundamental mode, actually a sine wave, during one period of vibration, the string achieves maximum tension – once up, once down – during this period.

This extension effect, and therefore an increase in string tension due to its elasticity, is occurring at twice the fundamental frequency, corresponding to the octave higher, 1^{st} overtone. Similarly, all other modes have increased tension effects at twice their respective frequencies. So, could this increased tension effect be reinforcing the 'higher sounding' perception? A rather extreme view of this, unsupported by analysis (Decker, 2007), is that the fundamental frequency is eliminated due to this effect, and the lowest observed frequency is the first harmonic.

The purpose of this paper is to study the effect of tension variation on the tonal response of a guitar by examining how the normal component of string tension acting on the bridge, i.e., the force that sets the guitar sound board in motion, is changed from the constant tension case. In short, what is the effect on the overtone structure?

The Model Assumptions

The approach to assessing the tension during vibration is that which has been used by all authors, viz., Hooke's law, applied to the increase in length during vibration. In the expression for length of the vibrating string profile, the value for the displacement function is taken to be that predicted by the classical linear theory. Longitudinal motion is ignored, as the speed of sound in a string, given by $\sqrt{\frac{E}{\rho}}$, (E = Young's modulus, ρ = density), is of the order of 2000 m/s for nylon, and very much higher for steel. As the string length of the tenor guitar is 0.65 m, it follows that tension changes are distributed virtually instantaneously throughout the string. Consequently, it is only necessary to consider temporal variations of the tension.

In making use of the known solution to the one-dimensional wave equation, the change in tension is estimated by assuming that the profile of the actual vibration is closely approximated by the profile found assuming a constant tension. The remaining assumptions of the proposed model, based on the one-dimensional wave equation, are given below.

(i) The string has spatially uniform mass density in the un-deflected state

(ii) The string is purely flexible in a lateral sense, i.e., there is neither elastic bending nor shear resistance to lateral motion, which motion is solely under the influence of the constant installed string tension.

(iii) For the string, there is no energy dissipation mechanism from either an internal source (heat generation), or externally from air damping.

(iv) The string is rigidly anchored at both ends.

(v) The lateral motion is very small compared to the vibrating length, so that the usual linearisation assumptions apply, i.e., self and cross-products of string deflection and slope may be ignored relative to first order terms in these quantities.

With regard to (i), modern methods of string manufacture ensure that it is a reasonable assumption for commercial strings, and is certainly very well justified if 'rectified' nylon strings are used.

For (ii), the inclusion of bending and shear effects is uncalled for as they affect only overtones well beyond those which have a major influence on perception, viz., the first half-dozen or so. Studies of this effect are due to Rayleigh (1894), Morse (1948), Young (1947), and many others.

In considering (iii), damping is neglected here, not because it is thought to be necessarily very small, but rather to determine an 'in principle' answer as to whether, in ideal conditions, the first overtone, and other higher overtones, can be significantly reinforced by incorporating an estimate of tension variation.

For (iv), the assumption of rigid anchoring would seem to be questionable. Whilst relative fixity at the nut is plausible, that at the bridge contradicts the basic mechanism of sound generation, viz., the component of string tension at the bridge, normal to the sound board, is what sets the sound board in motion, and hence the creation of sound waves. In kinematical terms, the vertical motion of the bridge is minute compared to the lateral displacement of the string, and so string profile is little affected, though the coupling to the mass-spring system of the sound board will tend to reduce, slightly, the overtone frequencies of the string.

For (v), this is implicitly accepted here, though the square of the displacement derivative is included in the expression for change of length.

The sequence of the proposed analysis is as follows:

(i) the analytical detail of the model is developed identifying the crucial parameters affecting the size of the normal force on the bridge, and the modal series that reflect the combined nature of the force arising from initial tension and tension increment;

(ii) then follows an analysis of the static force on the bridge, as this will indicate the relative magnitude of that due to initial tension and the tension increment to accommodate the lateral deflection by a point force from an original straight line; an important feature of this section is the use of the geometric stiffness concept to show why large deflections are inherently likely to be avoided when stopping the higher frets, thereby avoiding large tension increments;

(iii) the next step studies the question of how the tension increment affects the modes. A general result for the n^{th} mode is presented, but numerical assessment is only given for the first five, as these are considered to have most effect on tonal quality.

The Model

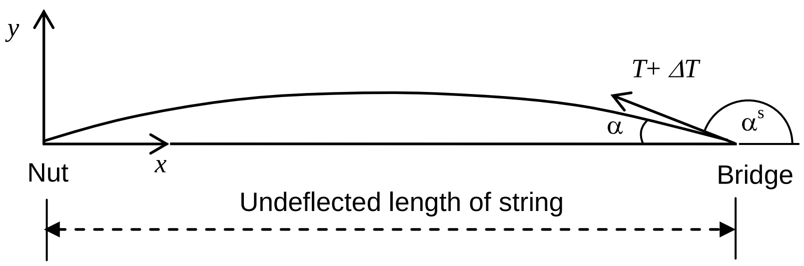


Fig. 4. String configuration

To discuss the contribution of string tension variation to the total force acting on the sound board we have, with reference to Fig. 4, the total vertical force as:

$$(T + \Delta T) sin\alpha, \tag{1}$$

where T is the initial static tension, ΔT the change in tension consequent on change of length during vibration, and α is the inclination of the string at some arbitrary time, to its un-deflected position, virtually parallel to the sound board.

Knowing the deflected profile of the string, $y(x,t)$, for the small deflections and slopes involved,

$$sin\alpha \approx -tan\alpha^{s} = -\frac{\partial y(l,t)}{\partial x} \tag{2}$$

Figure 5 shows a string of length l which is to be released from rest and initially deflected through a distance δ by a point force F acting at a distance a from the nut.

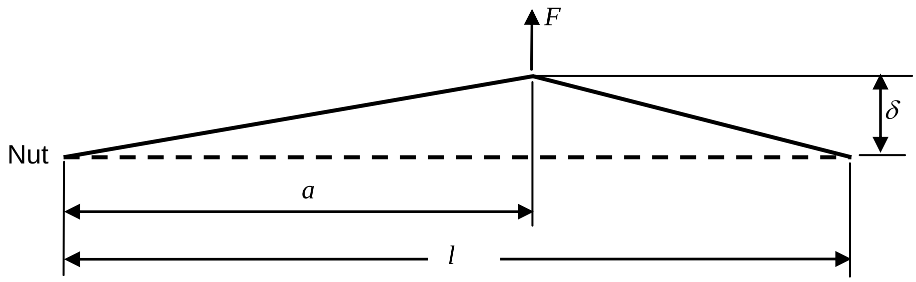


Fig. 5. String deflected by a plucking force

In this case, the profile of the string, according to the wave equation (Morse 1948: 87) is

$$y(l,t) = \frac{2\delta l^2}{\pi^2 a(l-a)} \sum_{n=1}^{\infty} \frac{sin\frac{na\pi}{l}}{n^2} sin\frac{n\pi x}{l} cos\frac{n\pi ct}{l} \tag{3}$$

where $c^2 = T/\rho$; c being the speed of propagation of lateral waves along the string, T the static tension, and ρ the lineal mass density (mass/unit length). It is helpful to write (3) a little more compactly by using:

$$r = \frac{a}{l}, \qquad \theta = \frac{a\pi}{l}, \qquad \omega = \frac{\pi c}{l}, \qquad \mu = \frac{\delta/l}{\pi r(1-r)},$$

so that

$$sin\alpha \approx = -\frac{\partial y(l,t)}{\partial x} = 2\mu \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} sin(n\theta)\, cos\,(n\omega t) = 2\mu S_1(\theta, \omega t) \tag{4}$$

The normal force component, $T\sin\alpha$, due to initial tension can be evaluated immediately. The change to initial tension, ΔT, from Hooke's law is

$$\Delta T = EA\frac{\Delta l}{l}, \tag{5}$$

where E is the Young's modulus of the string, and A is the cross-sectional area.

The change in length of the string is

$$\Delta l = \int_0^l \left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{\frac{1}{2}} dx - l \tag{6}$$

With the assumption of small slopes, (6) has the approximate value

$$\Delta l \cong \frac{1}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{7}$$

and substituting for $\frac{\partial y}{\partial x}$,

$$\Delta l \cong 2\mu^2 \int_0^l \left\{\sum_{n=1}^{\infty} sin\frac{n\theta}{n} cos\frac{n\pi x}{l} \cos n\omega t\right\}^2 dx \tag{8}$$

With the orthogonality of the cos $\frac{n\pi x}{l}$ over $(0,l)$, this reduces to

$$\Delta l \cong 2\mu^2 \sum_{n=1}^{\infty} \frac{sin^2 n\theta}{n^2} cos^2 n\omega t \int_0^l cos^2\frac{n\pi x}{l} dx = \mu^2 l \sum_{n=1}^{\infty} \frac{sin^2 n\theta}{n^2} \cos^2(n\omega t),$$

and so

$$\Delta T = EA\frac{\Delta l}{l} = EA\mu^2 \sum_{n=1}^{\infty}\frac{sin^2 n\theta}{n^2}cos^2 n\omega t = EA\mu^2 \sum_{n=1}^{\infty}\frac{sin^2 n\theta}{n^2}(1+\cos 2n\omega t) = \ EA\mu S_2(\theta,\omega t) \qquad (9)$$

Note, ΔT consists of non-oscillatory and oscillatory terms, the latter at double the frequency of the original modal frequencies.

Consequently,

$$(T+\Delta T)sin\alpha = 2\mu T S_1(\theta,\omega t) + \ 2EA\mu^3 S_1(\theta,\omega t)S_2(\theta,\omega t) \qquad (10)$$

Since $\mu \propto \frac{\delta}{l}$, and this ratio is very much less than one, it is to be expected that the second term, corresponding to tension increment, is likely to be much smaller than the basic $Tsin\alpha$ term.

The Static Assessment

Since the string will behave harmonically, the maximum variation in tension during oscillation cannot exceed that which occurs when the string is initially deflected. Consequently, the relative values of vertical force components $Tsin\alpha\,\big|_{t=0}$ and $\Delta Tsin\alpha\,\big|_{t=0}$ are important for assessing the significance of tension variation. Thus,

$$\{(T+\Delta T)sin\alpha\}\big|_{t=0} = 2\mu T S_1(\theta_1,0) + \ 2EA\mu^3 S_1(\theta_1,0)S_2(\theta_1,0)$$

$$= 2\mu T\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}sinn\theta + 2EA\mu^3\left(\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}sinn\theta\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2}sin^2 n\theta\right)$$

Both infinite series have finite summation formulae

$$\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}sinn\theta = \frac{\theta}{2}, \ \ -\pi<\theta<\pi; \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}sin^2 n\theta = \frac{\theta}{2}(\pi-\theta), \ \ 0<\theta<\pi$$

so that the expression for total force is just

$$2\mu T\frac{\theta}{2} + \ 2EA\mu^3\frac{\theta}{2}\left\{\frac{\theta}{2}(\pi-\theta)\right\}, \qquad 0<\theta<\pi$$

and as $\theta = r\pi$, this reduces to $r\pi\mu T + \frac{1}{2}\pi^3 r^2(1-r)\mu^3 EA$.

Substituting $\mu = \frac{\delta/l}{\pi r(1-r)}$ we get

$$\frac{\delta/l}{(1-r)}T + \frac{(\delta/l)^3}{2r(1-r^2)}EA \tag{11}$$

Note that $\frac{\delta/l}{1-r}$ is simply $\sin\alpha\,|_{t=0}$

It is now possible to evaluate this result, and judge the relative importance of tension increment. Since the first string carries a substantial amount of melodic line, it is chosen for evaluation. A typical first string is a D'Addario J4301, for which the makers quote a tension of 63.8 N and a cross-sectional area of 3.6×10^{-6} m^2. A typical Young's modulus for nylon is ~ 5×10^9 N/m^2.

The relative magnitudes of $T\sin\alpha$ and $\Delta T\sin\alpha$ are evaluated for two typical deflections, δ = 3 mm and 5 mm, as well as for one extreme case, δ = 10 mm, over the indicated plucking positions. This is done initially for the open string (Table 1).

	r / δ [mm]	1/2	2/3	3/4	4/5	5/6
$T\sin\alpha$ [N]	3	0.58892	0.88338	1.17785	1.47231	1.76677
	5	0.98154	1.4723	1.9631	2.4538	2.9446
	10	1.9631	2.9446	3.9262	4.9077	5.8892
$\Delta T\sin\alpha$ [N]	3	0.0070787	0.011994	0.018954	0.27651	0.038225
	5	0.032772	0.055303	0.087392	0.12801	0.17697
	10	0.26217	0.44242	0.69914	1.0241	1.4157
$\frac{\Delta T\sin\alpha}{T\sin\alpha}$ [%]	3	1.2	1.4	1.6	1.9	2.16
	5	3.3	3.8	4.5	5.2	6.01
	10	13.3	15.0	17.8	20.9	24.0

Table 1. Magnitudes of vertical components $T\sin\alpha$ and $\Delta T\sin\alpha$. Open string: l = 650mm

A similar table can be produced when the 12th fret is stopped and the same deflections are used; one simply multiplies the $T\sin\alpha$ entries by 2, and $\Delta T\sin\alpha$ by 8. Consequently, the $\Delta T\sin\alpha/T\sin\alpha$ percentages are multiplied by 4. Taking just the results corresponding to r = 5/6, the relevant percentages become, for δ = 3 mm: 8.64%, for δ = 5 mm: 24.04%, and for δ = 10mm: 96%. These are significant increases in the cases of δ = 3mm and 5 mm, and remarkable in the case of δ = 10mm. However, for much typical playing, they will not occur, for the reasons discussed below.

A lateral deflection of 10 mm is so large that it can only be produced by pulling, rather than plucking, i.e., it will not be produced in a normally sequenced set of notes. If it is produced, then the effect termed 'pitch glide' occurs, as noted earlier. While pitch glide, in principle, is always present, typical playing shows that an initially falling 'wow' effect, of short period, is only noticeable when the initial deflection is large, 10 mm, or more. Such large deflections, save in some *avant garde* compositions where pitch distortion is deliberately invoked, are avoided in order to maintain tonal quality. Further, except for relatively short periods to increase or decrease the level of sound, player's fingers will apply relatively constant levels of force. This fact has an important consequence.

Consider the balance of forces involved in plucking a string, as illustrated in Fig. 6.

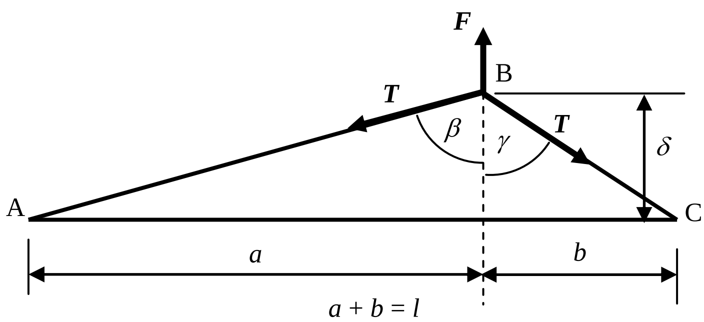


Fig. 6. Balance of forces in a plucked string

The plucking force, F, has to balance the string tensions in AB and BC. Assuming no friction at the point of force application, and ignoring the usually small elastic increase in tension

$$F = Tcos\beta + Tcos\gamma = T\left[\frac{\delta}{AB} + \frac{\delta}{BC}\right] \tag{12}$$

For small displacements, this will be closely approximated by

$$F = T\delta\left[\frac{1}{a} + \frac{1}{b}\right] \tag{13}$$

which can be re-written, on putting $a = rl$, $b = (1-r)l$,

$$F = \left[\frac{T}{r(1-r)l}\right]\delta \tag{14}$$

The entity $\frac{T}{r(1-r)l}$ is well known in the analysis of large architectural structures, such as tensioned cable nets, and is referred to as a 'geometric stiffness' (Lewis, 2003).

At the 12th fret, the vibrating length is halved, and if the point of force application is moved to obtain the same value of r as obtained for an open string, geometric stiffness is doubled, and for the same F, deflection is halved. Consequently, normal playing has an inherent tendency to avoid large deflections as the higher frets are broached. More precisely, the player's right hand usually remains relatively fixed as higher frets are stopped, or moves slightly to plucking at something like half the vibrating length. Taking the case of no movement of the right hand, there is the following remarkable result, given below.

With the usual assumptions of small deflection and constant plucking force, the factor μ,

$$\mu = \frac{\delta/l}{\pi r(1-r)} \tag{15}$$

can be shown (equations 16-19 below) to have the same value regardless of the fret stopped, if the position of force application remains fixed.

The frets raise the pitch by a semi-tone, and the vibrating length at the m^{th} fret from the nut is in equal temperament, $l/2^{m/12}$, $m = 1,2....$ With the same plucking position, r becomes r' where

$$r' = \frac{1}{l/2^{m/12}}\left[l/2^{m/12} - (1-r)l\right] = 2^{m/12}\left[r - \left(1 - \frac{1}{2^{m/12}}\right)\right] \tag{16}$$

with $\frac{1}{2} < r < 1$, always.

Deflection for this case can be calculated from

$$F = \left[\frac{T}{r'(1-r')l/2^{m/12}}\right]\delta' \tag{17}$$

and for the open string, from (14), viz.,

$$F = \left[\frac{T}{r(1-r)l}\right]\delta \tag{18}$$

On dividing (18) by (14) and re-arranging

$$\delta' = \frac{r'(1-r')}{2^{m/12}r(1-r)}\delta$$

and the value of μ, now μ', for the m^{th} fret, is

$$\mu' = \frac{\delta'/(l/2^{m/12})}{\pi r'(1-r')} = \frac{1}{\pi r'(1-r')}\frac{r'(1-r')}{2^{m/12}(1-r)}\frac{\delta}{l/2^{m/12}} = \frac{\delta/l}{\pi r(1-r)} \tag{19}$$

which is the value for the open string, plucked with position $a = rl$.

Of course, the result is approximate, but does imply that results for normal force on the bridge obtained for the open string will be indicative of results for the fretted case.

The Tension Increment Contribution to the Overtones

From (10), the vertical force on the bridge due to initial tension is

$$2\mu T S_1(\theta,\omega t) = 2\mu T \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n} sinn\theta cosn\omega t$$

so that the nodal component is

$$2\mu T\frac{(-)^{n+1}}{n} sinnrx, \quad n = 1,2 \ldots.. \tag{20}$$

The tension increment term arises from the product of two infinite series. As one of them, $S_2(\theta,\omega t)$, is absolutely convergent, the product of the two series can be found by multiplying them together term by term. Consequently

$$2EA\mu^3 TS_1(\theta,\omega t)S_2(\theta,\omega t) =$$

$$= 2EA\mu^3\left(\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}sinn\theta cosn\omega t\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2}sin^2n\theta cos^2n\omega t\right)$$

$$= 2EA\mu^3\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2q\theta cosp\omega t\, cos^2q\omega t \tag{21}$$

The temporal dependence, $\cos p\omega t\, cos^2q\omega t$, can be reduced to just cosine terms by the use of elementary trigonometric relations

$$\cos p\omega t\, cos^2 q\omega t = \frac{1}{2}\cos p\omega t(1+\cos 2q\omega t)$$

$$= \frac{1}{2}\cos p\omega t + \frac{1}{4}(\cos(2q+p)\omega t + \cos(2q-p)\omega t) \tag{22}$$

There is, therefore, a contribution to all modes as *p*,*q* go through all values, 1,2,..., arising from these terms. Leaving aside the coefficient, $2EA\mu^3$, the three contributions from (22), and noting (21), can be grouped into terms A, B, and C as follows.

A)

$$\frac{1}{2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2q\theta cosp\omega t$$

This gives a term contributing to the successive modes

$$\frac{1}{2}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2q\theta\ ,\ \ p=1,2,\ldots$$

B)

$$\frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2q\theta cos(2q-p)\omega t \tag{24}$$

With regard to (24), there is a term independent of time occurring when $2q - p = 0$, for then

$$\cos{(2q - p)\omega t} = \cos 0 = 1.$$

The non-oscillatory contribution $V_0(\theta)$ is, therefore,

$$V_0(\theta) = \frac{1}{4}\sum_{q=1}^{\infty}\frac{(-)^{2q+1}}{2q \cdot q^2} sin\ 2q\theta sin^2 q\theta \tag{25}$$

This term arises through the non-oscillatory term of ΔT (equation 9). This constant force does not contribute to the vibrating force on the bridge, but only a 'constant' deflection of the sound board. Of course, this 'constant' component is activated at the rate of plucking, so, sound board deflection is, in consequence, varying. However, the rate of string vibration is at least an order of magnitude higher than a typical fast rate of plucking, perhaps 5-10 per second, so the acoustic consequence is negligible.

Since cosine is an even function, there are modal contributions when $2q - p = \pm n$, n being a positive integer. Taking the case $2q - p = n$, and using the constraint $p, q \geq 1$, and starting from the lowest values of (p,q) that satisfy $2q - p = \pm n$, two sets of (p,q) emerge

If n is odd $\begin{cases} p = 2k - 1 \\ q = k + \dfrac{n-1}{2} \end{cases}$ $k = 1, 2,..,$

If n is even $\begin{cases} p = 2k \\ q = k + \dfrac{n}{2} \end{cases}$ $k = 1, 2,..,$

Similarly, when $2q - p = -n$, there are simpler results holding whether n is even or odd:

$$\begin{cases} p = 2k + n \\ q = k \end{cases} \qquad k = 1, 2,..,$$

The modal term,

$$\frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2 q\theta$$

therefore, consists of a sum of two series, which adopts a different form depending on whether *n* is odd or even:

n-odd

$$\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k+n)k^2}sin(2k+n)\theta sin^2 k\theta + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\left(k+\frac{n-1}{2}\right)^2}sin(2k-1)\theta sin^2\left(k+\frac{n-1}{2}\right)\theta \qquad (26)$$

n-even

$$-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k+n)k^2}sin(2k+n)\theta sin^2 k\theta - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{2k\left(k+\frac{n}{2}\right)^2}sin2k\theta sin^2\left(k+\frac{n}{2}\right)\theta \qquad (27)$$

where $n = 1,2,...$

C)

$$\frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{(-)^{p+1}}{pq^2}sinp\theta sin^2 q\theta cos(2q+p)\omega t \qquad (28)$$

As p, $q \geq 1$, this term can only contribute for $2q + p \geq 3$. This term is interesting in that the number of contributions to the modal sequence is finite for each mode, but this number gradually increases. As with B, $2q + p = n$, there is a different formula for *n*, odd or even. The pairs (q, p) are generated as follows:

$$n\text{-odd} \quad \begin{cases} q = k \\ p = n - 2k \end{cases} \qquad k = 1, 2,.., \tfrac{n-1}{2},$$

$$n\text{-even} \quad \begin{cases} q = k \\ p = n - 2k \end{cases} \qquad k = 1, 2,.., \tfrac{n-2}{2},$$

The modal terms are:

for *n*– odd:

$$\frac{1}{4}\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{(n-2k)k^2} sin(n-2k)\theta sin^2 k\theta \tag{29}$$

for *n*– even:

$$-\frac{1}{4}\sum_{k=1}^{\frac{n-2}{2}} \frac{1}{(n-2k)k^2} sin(n-2k)\theta sin^2 k\theta \tag{30}$$

For further reference, the terms for n = 3, 4, and 5 are noted

$$n = 3: \qquad \frac{1}{4}\cdot\frac{1}{1\cdot 1^2}\sin\theta sin^2\theta = \frac{sin^3\theta}{3} \tag{31}$$

$$n = 4: \qquad \frac{1}{4}\cdot\frac{(-1)}{2\cdot 1^2}\sin 2\theta sin^2\theta = -\frac{sin^3\theta\cos\theta}{4} \tag{32}$$

$$n = 5: \qquad \frac{1}{4}\cdot\frac{1}{3\cdot 1^2} sin\,3\theta sin^2\theta + \frac{1}{4}\cdot\frac{1}{1\cdot 2^2} sin\,\theta sin^2 2\theta = \frac{1}{2}sin^3\theta - \frac{7}{12}sin^5\theta \tag{33}$$

Combining the results of A), B), and C), gives modal terms, $V_n(\theta)$ defined below:

<u>n– odd:</u>

$$V_n(\theta) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{nk^2}\sin n\theta sin^2 k\theta + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k+n)k^2} sin(2k+n)\theta sin^2 k\theta$$
$$+\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\left(k+\frac{n-1}{2}\right)^2} sin(2k-1)\theta sin^2\left(k+\frac{n-1}{2}\right)\theta$$

plus when $n \geq 3$

$$+\frac{1}{4}\sum_{k=1}^{\frac{n-1}{2}}\frac{1}{(n-2k)k^2} sin(n-2k)\theta sin^2 k\theta \tag{34}$$

<u>n– even:</u>

$$V_n(\theta) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{nk^2}\sin n\theta sin^2 k\theta - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(2k+n)k^2} sin(2k+n)\theta sin^2 k\theta$$
$$-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{2k\left(k+\frac{n}{2}\right)^2} sin2k\theta sin^2\left(k+\frac{n}{2}\right)\theta$$

plus when $n > 3$

$$-\frac{1}{4}\sum_{k=1}^{\frac{n-2}{2}}\frac{1}{(n-2k)k^2} sin(n-2k)\theta sin^2 k\theta \tag{35}$$

While some of the series in (26) to (35) can be shown to have closed forms, most were not readily so reduced. Therefore, the series were evaluated numerically in 15 significant figure precision using Microsoft Excel, employing the first twenty terms of each series. This was done for each of the nominal plucking positions, $l/2$, $2l/3$, $3l/4$, $4l/5$, $5l/6$.

The Modal Contributions of the Initial Tension and Tension Increment to the Normal Force on the Bridge

The Modal Contributions from the Initial Tension, T

The contribution to the initial tension is given by (20)

$$2\mu T\left\{\frac{(-)^{n+1}}{n}sinnr\pi\right\} \quad n = 1,2 \ldots\ldots$$

where $\mu = \frac{\delta/l}{\pi r(1-r)}$, r = 1/2, 2/3, 3/4, 4/5, 5/6, T = 63.8 N, δ = 3 mm, or 5 mm, l = 650 mm throughout.

The calculated modal terms for varying values of *r* and *n* are given in Table 2.

n \ *r*	1/2	2/3	3/4	4/5	5/6
1	1	0.96603	0.70711	0.58779	0.50000
2	0.0	0.43301	0.50000	0.47553	0.43301
3	-0.33333	0.0	0.23570	0.31702	0.33333
4	0.0	-0.21561	0.0	0.14695	0.21651
5	0.20000	-0.17321	-0.14142	0.0	0.10000

Table 2. The modal terms for the case of initial tension

The scaling factors converting the modal terms to normal force are $2\mu T$. These give modal contributions to normal force from the initial tension, as shown in Tables 3a and 3b.

r / n	1/2	2/3	3/4	4/5	5/6
1	0.74984	0.73056	0.70696	0.68867	0.67485
2	0.0	0.36527	0.49990	0.55715	0.58443
3	-0.24994	0.0	0.23565	0.37143	0.44990
4	0.0	-0.18264	0.0	0.17217	0.29222
5	0.14997	-0.14611	-0.14139	0.0	0.13497

Table 3a. Modal contribution to normal force (in Newtons) from the initial tension; δ = 3 mm

It can be seen that the contribution to normal force (measured in Newtons) from the initial tension is highest for the fundamental mode of vibration (n=1) and gets progressively lower for the higher modes.

r / n	1/2	2/3	3/4	4/5	5/6
1	1.2497	1.2176	1.1783	1.1478	1.1248
2	0.0	0.60878	0.83317	0.92858	0.97405
3	-0.41657	0.0	0.39275	0.	0.74983
4	0.0	-0.30440	0.0	0.28695	0.48703
5	0.24997	-0.24352	-0.23565	0.0	0.22495

Table 3b. Modal contribution to normal force (in Newtons) from the initial tension; δ = 5mm

The Modal Contributions to Normal Force from the Tension Increment ΔT

As shown in (10), (34) and (35), the modal contributions to normal force from tension increment are given by $2EA\mu^3 V_n(\theta)$, where $V_0(\theta)$ is given by (25), and $V_n(\theta)$ by (34) and (35).Results for values of $V_n(\theta)$ are presented in Table 4.

n \ r	1/2	2/3	3/4	4/5	5/6
0	0.0	-0.09567	-0.1211	-0.1241	-0.1196
1	0.7498	0.6894	0.5572	0.4565	0.3794
2	0.0	0.2169	0.2524	0.2222	0.1841
3	0.8720	0.1745	0.2217	0.2081	0.1787
4	0.0	-0.0348	0.0491	0.0968	0.1084
5	0.03141	-0.03177	0.01001	0.06283	0.08604

Table 4. Modal contributions $V_n(\theta)$ from the tension increment; n = 0 corresponds to non-oscillatory component $V_0(\theta)$

The scaling factors $2EA\mu^3$convert the modal terms to normal force given in Tables 5a and 5b.

n \ r	1/2	2/3	3/4	4/5	5/6
0	0.0	-0.00099544	-0.0020974	-0.0034571	-0.0050965
1	0.0057364	0.0071711	0.0096490	0.012722	0.016165
2	0.0	0.0022562	0.0043708	0.0061925	0.0078438
3	0.00063689	0.0018151	0.0038391	0.0057995	0.0076120
4	0.0	-0.00036199	0.00085026	0.0026977	0.0046185
5	0.00029412	-0.00033047	0.00017331	0.0017507	0.0036580

Table 5a. Modal contributions to normal force (in Newtons) due to the tension increment; δ = 3 mm; n = 0 corresponds to non-oscillatory component $V_0(\theta)$

n \ r	1/2	2/3	3/4	4/5	5/6
0	0.0	-0.0046052	-0.0097102	-0.016005	-0.023595
1	0.026557	0.033111	0.044671	0.058898	0.074838
2	0.0	0.015078	0.020235	0.028667	0.036314
3	0.0029486	0.0084032	0.0177736	0.026850	0.035241
4	0.0	-0.0016759	0.039364	0.012489	0.021357
5	0.0013667	-0.0015210	0.00080236	0.0081051	0.016967

Table 5b. Modal contributions to normal force (in Newtons) due to the tension increment; δ = 5 mm; n = 0 corresponds to non-oscillatory component $V_0(\theta)$

It can be seen that modal contributions to normal force due to the tension increment in both cases (δ = 3mm and δ = 5 mm) are extremely small compared to the case of initial tension (Tables 3a and 3b).

The crucial items for tonal response are the changes in the ratios of normal modal force, compared to those occurring with just initial tension, i.e., the ratios:

$$\lambda_T = \frac{(Tsin\alpha)_n}{(Tsin\alpha)_1}, \quad \lambda_{T+\Delta T} = \frac{\{(T+\Delta T)sin\alpha\}_n}{\{(T+\Delta T)sin\alpha\}_1}$$

are given in Tables 6, 7a, and 7b.

r / n	1/2	2/3	3/4	4/5	5/6
2	0.0	0.500	0.707	0.809	0.866
3	-0.333	0.0	0.333	0.539	0.666
4	0.0	-0.250	0.0	0.250	0.433
5	0.200	-0.200	-0.200	0.0	0.200

Table 6. Values of $\lambda_T = (Tsin\alpha)_n/(Tsin\alpha)_1$

r / n	1/2	2/3	3/4	4/5	5/6
2	0.0	0.499	0.704	0.803	0.857
3	-0.330	0.002	0.334	0.538	0.662
4	0.0	-0.248	0.002	0.249	0.429
5	0.199	-0.198	-0.19	0.002	0.119

Table7a. Values of $\lambda_{T+\Delta T} = \{(T+\Delta T)\sin\propto\}_n/\{(T+\Delta T)\sin\propto\}_1$;$\delta$= 3 mm

r / n	1/2	2/3	3/4	4/5	5/6
2	0.0	0.523	0.698	0.818	0.842
3	-0.324	0.007	0.339	0.370	0.654
4	0.0	-0.244	0.003	0.248	0.423
5	0.197	-0.193	-0.192	0.007	0.201

Table 7b. Values of $\lambda_{T+\Delta T} = \{(T+\Delta T)\sin\propto\}_n/\{(T+\Delta T)\sin\propto\}_1$; δ = 5 mm

The other crucial parameter is $\frac{\{(T+\Delta T)\sin\propto\}_1}{\{T\sin\propto\}_1} = \lambda$, given in Table 8.

r	1/2	2/3	3/4	4/5	5/6
λ					
(δ = 3 mm)	1.0	1.0104	1.0136	1.0185	1.024
(δ = 5 mm)	1.0213	1.0273	1.0379	1.0513	1.045

Table 8. The ratio $\lambda = \{(T + \Delta T) \sin \propto\}_1 / \{T \sin \propto\}_1$; δ = 3 mm, and 5 mm

Analysis of results

The results in Table 2 give the modal contributions from just the initial tension. The results in Tables 3a and 3b give the corresponding modal contributions to the normal force on the bridge. It can be seen that the first three modes are generally more significant. With plucking positions at a = 4/5l and 5/6l , it is clear that while the fundamental mode is still dominant, the fall off in contributions is much less marked, i.e., modes 2 to5 are making a greater contribution. The tonal consequence is clear; the sound is 'brighter'.

The results in Table 4, giving modal contributions from the tension increment, show that the entries are mostly smaller, around 50%-70%, of the corresponding results for the initial tension model (Table 2). When these are converted to normal force contributions (Tables 5a, 5b), they are typically less than 10% of those from the relevant mode of the initial tension model (Tables 3a and 3b). As one would expect, the results from the tension increment all add to those for the initial tension, so, the normal force on the bridge is slightly increased. This leads to increased sound board amplitudes and, therefore, greater sound pressure level. Table 6 gives the ratio of the normal force contribution from the overtones, n = 2 to 5, to that from the fundamental mode (n = 1), for the case of initial tension, T. These values are to be compared with those in Tables 7a and 7b, for the total tension $T+\Delta T$. Concentrating on the more usual playing positions, at a = 4/5l and 5/6l , it is clear that while all the modal contributions are increased by the tension increment, the principal effect is on the fundamental mode, so much so, that the ratio of overtone contributions to normal force are all reduced relative to the results for just initial tension.

Finally, results in Table 8, which exhibit the ratios of total tension in mode 1 to initial tension resulting from mode 1, show just how small the tension increment effect is (less than 5%).

Conclusions

Maximum tension variation occurs twice during the fundamental period, i.e., at precisely twice the frequency of the fundamental mode, and is, therefore, coincident with its first overtone. While the proposal that tension variation could substantially affect the overtone response is plausible, the results of the proposed model show no support for this conjecture. Indeed, while all modal contributions (due to tension variation) are increased and, therefore, some brightness ensues, it is the fundamental mode that is mainly enhanced. The perturbation model shows why this is so. Firstly, the tension increment, via the elementary trigonometric device, decomposes to terms carrying the same harmonic factors, $\cos n\omega t,\quad n = 1,2,\ldots,$ as in the classical, constant tension model. Thus, the string has the same frequency content as the constant tension model, but with increased contribution to the normal force on the bridge. Secondly, what prevents the tension increment from making a significant input to tonal response is the smallness of the factor $2EA\mu^3$ compared to $2\mu T$. If f is the dynamic force on the bridge, then it has a form: $f = \sum_{n=1}^{\infty} f_n \cos n\omega t$, where f_n is the force contribution from the n^{th} mode. The tension increment changes the f_n to f'_n, say, though the differences are small. Consequently, the resulting graph of f will look much the same as the classical one, but covering a slightly wider spread.

Even though the significance of tension increment can be increased by considering large initial deflections, practical considerations of the plucking mechanism and the need to control tone quality, i.e., avoid pitch glide – save in certain rare circumstances – limit this option. Large deflections while playing in the higher frets could lead to significant tension increments, but, again, the practical observation that plucking force (approximately constant), combined with the novel result stemming from the use of geometric stiffness, that $\mu = \frac{\delta/l}{\pi r(1-r)}$ is constant for constant plucking position, shows that the effects of the initial tension and tension increment remain unchanged regardless of whether one is playing in the lower, or higher, frets.

In summary, the results of the mathematical model presented in this article show no support for the conjecture that string tension variation during vibration has a significant effect on the overtone response.

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